INTEGRAL EQUATIONS OF EQUILIBRIUM OF THIN ELASTIC CYLINDRICAL SHELLS

(INTEGRAL'NYE URAVNENIIA RAVNOVESIIA TONKYKH UPRUGIKH TSILINDRICHESKIKH OBOLOCHEK)

PMM Vol.23, No.3, 1959, pp. 611-615

N.I. REMIZOVA
(Kiev)

(Received 27 January 1958)

In his contributions [1-3] to the theory of shells, Kil'chevskii derives a general method of solving the static problem of the theory of shells by reducing this problem to the treatment of a certain system of integral equations. In the following we give the results of further development of his method with reference and application to the case of cylindrical shells.

The system of the integro-differential equations of equilibrium of a cylindrical shell is obtained on the basis of the theorem of work reciprocity (Theorem of Betti [4]). According to this Theorem we consider, in the well-known manner, two systems of forces and displacements: the first system consists of prescribed forces and sought displacements, the second one consists of auxiliary forces and auxiliary displacements.

We treat the cylindrical shell as a continuous three-dimensional medium. The middle surface of the shell is used as coordinate surface, and the position of a point on this surface is determined by the coordinates x and s; these are the distance along a generator and the length of the arc of the directing curve, respectively (see Fig. 1). The third coordinate z is the distance, measured along the normal, between the point $M(x, s)$ of the middle surface and a point considered; z varies between the limits $-1/2\,h$ and $+1/2\,h$, where h is the constant thickness of the shell.

Assume that a concentrated unit force, acting in the direction e_j of the local reference coordinates, is applied to the middle surface of the shell at the point $N(x_N, s_N)$ of that surface. The component of the linear displacement, produced by this force at an arbitrary point $M(x, s)$, in the direction e_a shall be denoted by $u_{(j)a}(M; N)$. Here and in the following the subscript in the parentheses indicates the direction of the force. The angle of rotation of the normal to the middle surface at the point M around e_γ shall be denoted by $\omega_{(j)\gamma}(M; N)$. The stress resultants and stress couples at an arbitrary point $L(x_L, s_L)$ of the boundary contour shall be denoted by $T_{(j)\alpha}(L; N)$ and $L_{(j)\alpha}(L; N)$, respectively. The subscripts α, j assume the values 1, 2, 3; the subscript γ assumes the values 1, 2. This will be our basic system of loads and displacements.

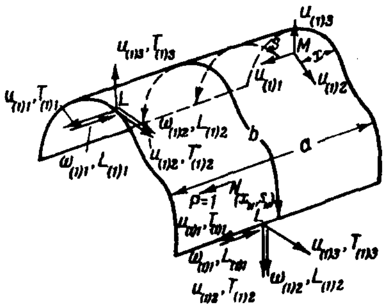

Fig. 1.

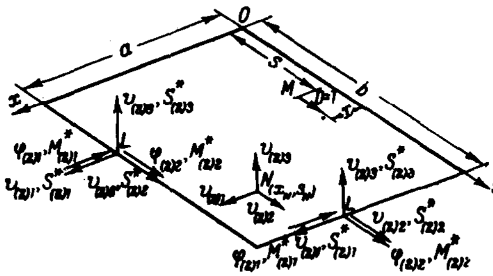

Fig. 2.

Now imagine the cylindrical shell rolled out into a plane and consider the resulting rectangular plate (Fig. 2). Assume a concentrated unit force parallel to e_α to be applied to the plate at the point $M(x, s)$ of the latter. The linear and angular displacements, arising at an arbitrary point $N(x_N, s_N)$ of the plate as a result of the action of that force, shall be denoted by $v_{(\alpha)\beta}(N; M)$, $\phi_{(\alpha)\gamma}(N; M)$, respectively, while the stress resultants and stress couples acting along the boundary line shall be denoted by $S^*_{(\alpha)\beta}(L; M)$, $M^*_{(\alpha)\gamma}(L; M)$, respectively, where $\beta = 1, 2, 3$.

Let us impose the displacements $v_{(\alpha)\beta}$ upon the points of the middle surface of the shell.

In order to produce these displacements in the shell, we have to apply to the shell some loads distributed over the curved as well as the boundary surfaces in addition to the concentrated unit force. This additional loading is replaced, in the theory of thin shells, by the following loadings:

(a) the loading $K_{(\alpha)\beta}(Q; M)$ and the moments $G_{(\alpha)\gamma}(Q; M)$, applied to the points of the middle surface;

(b) the auxiliary stress resultants $S_{(\alpha)\beta}(L; M)$ and stress couples $M_{(\alpha)\gamma}(L; M)$, applied to the boundary contour of the middle surface of the shell. The stress resultants $S_{(\alpha)\beta}$ and the stress couples $M_{(\alpha)\gamma}$ differ from the corresponding quantities of the plate by terms, which disappear together with the curvature of the shell. The displacements $v_{(\alpha)\beta}$ and the loadings necessary to produce them in the shell will be used as our auxiliary system.

The theorem of work reciprocity, if applied to our basic and auxiliary systems of forces and displacements, permits to write down the following system of integro-differential equations:

$$u_{(j)\alpha}(M;N) = v_{(\alpha)j}(N;M) - \int_0^a\int_0^b H_{(\alpha)\beta}(Q;M)\,u_{(j)\beta}(Q;N)\,dx_Q\,ds_Q + A_{(j)\alpha}(M;N)$$

$$(\alpha,\ \beta,\ j = 1,2,3) \qquad (1)$$

where

$$H_{(\alpha)1} = K_{(\alpha)1}, \qquad H_{(\alpha)2} = K_{(\alpha)2} + kG_{(\alpha)1}$$

$$H_{(\alpha)3} = K_{(\alpha)3} + \frac{\partial}{\partial x}G_{(\alpha)2} + \frac{\partial}{\partial s}G_{(\alpha)1}$$

$$A_{(j)\alpha}(M;N) = \int_0^a [T_{(j)\beta}(L;N)\,v_{(\alpha)\beta}(L;M) + L_{(j)\gamma}(L;N)\,\varphi_{(\alpha)j}(L;M) - G_{(\alpha)1}(L;M)\,u_{(j)3}(L;N) - S_{(\alpha)\beta}(L;M)\,u_{(j)\beta}(L;N) - M_{(\alpha)\gamma}(L;M)\,\omega_{(j)\gamma}(L;N)]_{s_L=0}^{s_L=b}\,dx_L + \int_0^b [T_{(j)\beta}(L;N)\,v_{(\alpha)\beta}(L;M) + L_{(j)\gamma}(L;N)\,\varphi_{(\alpha)\gamma}(L;M) - G_{(\alpha)2}(L;M)\,u_{(j)3}(L;N) - S_{(\alpha)\beta}(L;M)\,u_{(j)\beta}(L;N) - M_{(\alpha)\gamma}(L;M)\,\omega_{(j)\gamma}(L;N)]_{x_L=0}^{x_L=a}\,ds_L$$

$$(\alpha,\ \beta,\ j = 1,2,3;\ \gamma = 1,2)$$

while $k = k(s)$ denotes the principal curvature of the middle surface of the shell.

The equations (1) represent the fundamental system of integro-differential equations of equilibrium of cylindrical shells. Determination of the nine functions $u_{(j)\alpha}(M;N)$ leads to Green's tensor, which permits to find the displacements produced by the arbitrary loading.

The system (1) shows that the desired displacements represent sums of two terms. The first term is the corresponding displacement of a point at the middle plane of the plate. The second term, containing an integral over the middle surface of the shell and its boundary line, expresses the general influence of the shell curvature and of the difference of the boundary conditions of plate and shell on the displacements of the points of the middle surface of the shell. Consequently, the best version of the auxiliary system is the one derived from the solution of the problem of the plate under the action of a concentrated unit force, with boundary conditions which are identical to those of the shell. In the worst case, if the solution of the corresponding problem of the plate is difficult, the auxiliary displacements are to be represented by a sum of the form

$$v_{(\alpha)\beta} = V_{(\alpha)\beta} + v'_{(\alpha)\beta} \qquad (2)$$

where the $V_{(\alpha)\beta}$ are functions possessing a singularity corresponding to the action of a concentrated unit force on a plate, to be determined by solving the known problems: the one of the two-dimensional theory of elasticity [4] and the other of the theory of plates in bending [5]; $v'_{(\alpha)\beta}$ are arbitrary functions, regular on the middle surface of the shell. The functions $v'_{(\alpha)\beta}$ determine such displacements of the points at the middle surface of the shell, which are produced by some non-concentrated forces and which are not necessarily solutions of the homogeneous equations of the problems indicated above; these functions are being introduced for the purpose of satisfying the boundary conditions of the shell. It will become evident in the following that the displacements $v'_{(\alpha)\beta}$ produce a change in the structure of the equations (1).

We shall determine the roots $H_{(\alpha)\beta}$ of the system (1) of equations by means of the differential equations of equilibrium for an element of the middle surface of the cylindrical shell. These equations are, in their general form

$$R_i^{\circ}(v_{(\alpha)\beta}) + R_i^{k}(v_{(\alpha)\beta}) + H_{(\alpha)i} = 0 \qquad (\alpha,\ \beta,\ i = 1,\ 2,\ 3) \tag{3}$$

where R_i^0 and R_i^k are homogeneous linear operators for the displacements; the first of these operators refers to the plate, while the second vanishes together with the curvature of the shell; the $H_{(\alpha)i}$ represent the components of the external load.

If the auxiliary displacements satisfy the corresponding boundary conditions of the plate, then the roots of the equations (1) are determined by the operator $R_i^k(v_{(\alpha)\beta})$, but if the displacements are chosen in the form (2), then the roots $H_{(\alpha)i}$ are determined by the sum

$$H_{(\alpha)i} = -R_i^{\circ}(v'_{(\alpha)\beta}) - R_i^{k}(v_{(\alpha)\beta})$$

Let us apply the results just obtained to the solution of the problem of a circular cylindrical shell acted upon by a concentrated normal force P and hinged along its edges.

If the auxiliary displacements are available in the form of double trigonometric series, then the three components of displacement of a point at the middle surface can be represented by means of closed expressions.

The displacements produced by a concentrated normal unit force in a plate with movable hinged supports are of the form

$$v_{(3)1} = v_{(3)2} = 0$$

$$v_{(3)3}(N;\ M) = \alpha_3 \sum_{mn} C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi s}{b} \sin\frac{m\pi x_N}{a} \sin\frac{n\pi s_N}{b} \tag{4}$$

where

$$\alpha_8 = \frac{4a^2\lambda}{\pi^4 D}, \qquad C_{mn} = (m^2 + \lambda^2 n^2)^{-2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}, \qquad \lambda = \frac{a}{b}$$

where E is the modulus of elasticity of first kind and ν is Poisson's ratio.

We consider here in some detail the solution of the plane problem of the theory of elasticity by means of double trigonometric series.

The required displacements, produced by the concentrated unit force directed along the α-th coordinate line satisfy the system of equations

$$\begin{aligned} &\frac{\partial^2 v_{(\alpha)1}}{\partial x_N^2} + \frac{1+\nu}{2}\frac{\partial^2 v_{(\alpha)2}}{\partial x_N \partial s_N} + \frac{1-\nu}{2}\frac{\partial^2 v_{(\alpha)1}}{\partial s_N^2} = q\delta_{(\alpha)1} \\ &\frac{1-\nu}{2}\frac{\partial^2 v_{(\alpha)2}}{\partial x_N^2} + \frac{1+\nu}{2}\frac{\partial^2 v_{(\alpha)1}}{\partial x_N \partial s_N} + \frac{\partial^2 v_{(\alpha)2}}{\partial s_N^2} = q\delta_{(\alpha)2} \qquad (\alpha = 1, 2) \end{aligned} \tag{5}$$

where

$$\delta_{(\alpha)k} = \begin{cases} 1 \text{ when } \alpha = k \\ 0 \text{ when } \alpha \neq k \end{cases}, \qquad q = \begin{cases} -(1-\nu^2)/Eh & \text{when } N = M \\ 0 & \text{when } N \neq M \end{cases}$$

If the solution of the system (5) is to be obtained in the form of double trigonometric series, then we have

$$\begin{aligned} v_{(1)1}(N; M) &= \alpha_1 \sum_{mn} a_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi s}{b} \cos\frac{m\pi x_N}{a} \sin\frac{n\pi s_N}{b} \\ v_{(1)2}(N; M) &= \alpha_2 \sum_{mn} b_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi s}{b} \sin\frac{m\pi x_N}{a} \cos\frac{n\pi s_N}{b} \\ v_{(2)1}(N; M) &= v_{(1)2}(M; N) \\ v_{(2)2}(N; M) &= \alpha_1 \sum_{mn} a^\circ_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi s}{b} \sin\frac{m\pi x_N}{a} \cos\frac{n\pi s_N}{b} \end{aligned} \tag{6}$$

where

$$\alpha_1 = \frac{4(1-\nu^2)\lambda}{\pi^2 Eh}, \qquad \alpha_2 = -\frac{4(1+\nu)^2\lambda^2}{\pi^2 Eh}$$

$$a_{mn} = \frac{m^2 + 2/(1-\nu)^{\lambda^2 n^2}}{(m^2+\lambda^2 n^2)^2}, \qquad b_{mn} = \frac{mn}{(m^2+\lambda^2 n^2)^2}$$

$$a^\circ_{mn} = \frac{1}{(m^2+\lambda^2 n^2)^2}\left(\lambda^2 n^2 + \frac{2}{1-\nu} m^2\right)$$

If the auxiliary displacements are chosen in the form (4) and (6), then the system of the integro-differential equations (1) assumes, in the case of the problem stated above, the form [6]

$$u_1(M;N) = -\int_0^a\int_0^b H_{(1)3}(Q;M)\,u_3(Q;N)\,dx_Q\,ds_Q$$

$$u_2(M;N) = -\int_0^a\int_0^b H_{(2)3}(Q;M)\,u_3(Q;N)\,dx_Q\,ds_Q \tag{7}$$

$$u_3(M;N) = v_{(3)3}(N;M) - \int_0^a\int_0^b [H_{(3)1}(Q;M)\,u_1(Q;N) +$$
$$+ H_{(3)2}(Q;M)\,u_2(Q;N) + H_{(3)3}(Q;M)\,u_3(Q;N)]\,dx_Q\,ds_Q$$

The roots $H_{(\alpha)\beta}$ of the system (7) are determined on the basis of the differential equations of equilibrium of the general technical bending theory of thin shells developed by Vlasov [7]. So we have

$$H_{(1)3}(Q;M) = \beta_1 \sum_{mn} A_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi s}{b} \sin\frac{m\pi x_Q}{a} \sin\frac{n\pi s_Q}{b}$$

$$H_{(2)3}(Q;M) = \beta_2 \sum_{mn} B_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi s}{b} \sin\frac{m\pi x_Q}{a} \sin\frac{n\pi s_Q}{b} \tag{8}$$

$$H_{(3)1}(Q;M) = \gamma_1 \sum_{mn} \alpha_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi s}{b} \cos\frac{m\pi x_Q}{a} \sin\frac{n\pi s_Q}{b}$$

$$H_{(3)2}(Q;M) = \gamma_2 \sum_{mn} \beta_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi s}{b} \sin\frac{m\pi x_Q}{a} \cos\frac{n\pi s_Q}{b}$$

$$H_{(3)3}(Q;M) = \gamma_3 \sum_{mn} \gamma_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi s}{b} \sin\frac{m\pi x_Q}{a} \sin\frac{n\pi s_Q}{b} \tag{9}$$

where

$$\beta_1 = \frac{4k\lambda}{\pi a}, \qquad A_{mn} = \frac{m(\lambda^2 n^2 - \nu m^2)}{(m^2 + \lambda^2 n^2)^2}$$

$$\beta_2 = \frac{4k\lambda^2}{\pi a}, \qquad B_{mn} = \frac{n\{m^2[\nu(1+\nu) - 2]/(1-\nu) - \lambda^2 n^2\}}{(m^2 + \lambda^2 n^2)^2}$$

$$\gamma_1 = -\frac{48a\lambda\nu k}{\pi^3 h^2}, \qquad \gamma_2 = -\frac{48a\lambda^2 k}{\pi^3 h^2}, \qquad \gamma_3 = \frac{48a^2k^2\lambda}{\pi^4 h^2},$$

$$\alpha_{mn} = \frac{m}{(m^2 + \lambda^2 n^2)^2}, \qquad \beta_{mn} = \frac{n}{(m^2 + \lambda^2 n^2)^2}, \qquad \gamma_{mn} = \frac{1}{(m^2 + \lambda^2 n^2)^2}$$

Eliminating u_1 and u_2 from the third equation with the aid of the first two equations, we can reduce the system (7) of integral equations to the one integral equation

$$u_3(M;N) = Pv_{(3)3}(M;N) - \int_0^a\int_0^b F_{(3)3}(Q;M)\,u_3(Q;N)\,dx_Q\,ds_Q \tag{10}$$

where

$$F_{(3)3}(Q;\,M) = H_{(3)3}(Q;\,M) - \int_0^a\!\!\int_0^b H_{(3)\beta}(P;\,M)\,H_{(\beta)3}(Q;\,P)\,dx_p\,ds_p \quad (\beta = 1,\,2)$$

The solution of the equation (10) can be written in the form

$$u_3(M;\,N) = \sum_{mn}\left[P\alpha_3 C_{mn}\sin\frac{m\pi x_N}{a}\sin\frac{n\pi s_N}{b} + L_{mn}(N)\right]\sin\frac{m\pi x}{b}\sin\frac{n\pi s}{b} \qquad (11)$$

where $L_{mn}(N)$ are unknown coefficients. Substituting (11) into (10) and making use of the orthogonality of the trigonometric functions in the interval considered, we obtain an equation for the determination of $L_{mn}(N)$.

Ultimately we get

$$u_3(M;\,N) = \frac{Pa^2}{D}\,[u_0(M;\,N) - u_k(M;\,N)] \qquad (12)$$

where

$$u_0 = \frac{4\lambda}{\pi^4}\sum_{mn}\frac{1}{(m^2+\lambda^2n^2)^2}\sin\frac{m\pi x_N}{a}\sin\frac{n\pi s_N}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi s}{b}$$

$$u_k = \frac{4\lambda\mu}{\pi^4}\sum_{mn}\frac{m^4}{(m^2+\lambda^2n^2)^2\,[(m^2+\lambda^2n^2)^4+\mu m^4]}\sin\frac{m\pi x_N}{a}\sin\frac{n\pi s_N}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi s}{b}$$

$$\mu = \frac{12\,(1-\nu^2)}{\pi^4}\,\frac{a^4}{R^2h^2}$$

From the relations (8) and (12) we find the quantities $H_{(1)3}$, $H_{(2)3}$ and u_3; substituting the results into the first two equations of the system (7) and carrying out the quadratures, we find

$$u_1(M;\,N) = \delta\sum_{mn}K_{mn}\sin\frac{m\pi x_N}{a}\sin\frac{n\pi s_N}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi s}{b} \qquad (13)$$

$$u_2(M;\,N) = \delta\sum_{mn}N_{mn}\sin\frac{m\pi x_N}{a}\sin\frac{n\pi s_N}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi s}{b} \qquad (14)$$

where

$$\delta = \frac{4P\lambda\mu R}{\pi Eha}, \qquad K_{mn} = \frac{m\,(\nu m^2 - \lambda^2n^2)}{\Delta_{mn}}, \qquad N_{mn} = \frac{n\,[m^2(2+\nu) + \lambda^2n^2]}{\Delta_{mn}}$$

$$\Delta_{mn} = (m^2 + \lambda^2n^2)^4 + \mu m^4$$

The solution obtained coincides completely with the results given for the same problem in Vlasov's book [7].

Thus we arrive at the following conclusion: if, in setting up the differential and integral equations, for the equilibrium of one and the same shell, we start from the same assumptions, then the two sets of results are identical.

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Translated by I.M.